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New approach to the quantum non-linear Schrödinger equation

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Abstract. A new technique is proposed which allows us to avoid the space- and UV-cutoffs when quantising the field-theoretic models integrable by means of the quantum spectral transform (R-matrix) method. The technique is based on a new definition of the monodromy matrix for the infinite interval. For the non-linear Schrödinger equation (repulsive case, zero density) the known excitation spectrum is reproduced.

1. Introduction

The quantisation of integrable field-theoretic models in the framework of the quantum spectral transform method (QSTM), see [1–3], is performed usually in a complicated, many-stage way including transition to a discrete model in finite volume and subsequently removing the ultraviolet and space cutoffs, cf the sine-Gordon model [1]. Rare exclusions are provided by the models having a ferromagnetic vacuum for which the technique using the monodromy matrix $\mathbf{T}(u)$ on the infinite interval is elaborated within QSTM [2, 3]. The technique allows us to investigate immediately continuous models in infinite volume, e.g. the non-linear Schrödinger equation [3] and the continuous su(1,1)-magnet [4]. It is based on the commutation relations

$$[\mathbf{A}(u), \mathbf{A}(v)] = [\mathbf{B}(u), \mathbf{B}(v)] = 0 \tag{1a}$$

$$\mathbf{A}(u)\mathbf{B}(v) = \tau(u-v)\mathbf{B}(v)\mathbf{A}(u) \tag{1b}$$

between the matrix elements of $\mathbf{T}(u)$

$$\mathbf{T}(u) = \begin{pmatrix} \mathbf{A}(u) & \mathbf{B}(u) \\ \mathbf{C}(u) & \mathbf{D}(u) \end{pmatrix}. \tag{2}$$

The relations (1) combined with the condition that the vacuum $|0\rangle$ is an eigenstate of $\mathbf{A}(u)$

$$\mathbf{A}(u)\,|0\rangle = |0\rangle$$

allow us to interpret $\mathbf{A}(u)$ as a generating function of commuting quantum integrals of motion and $\mathbf{B}(u)$ as a creation operator for excitations. Unfortunately, the algebra of matrix elements of $\mathbf{T}(u)$ is ill defined due to a singularity in the commutator $[\mathbf{C}(u), \mathbf{B}(v)]$;

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see [3]. In addition, for the majority of interesting models, the technique does not work. The reason is that the coefficient $\tau(u-v)$ in the commutation relation (1*b*) depends on the excitation spectrum of the system which, in fact, is to be determined [1].

It is possible to break the vicious circle by means of the new approach proposed in the present paper. The approach is based on replacing the matrix $\mathbf{T}(u)$ corresponding to the transition from $-\infty$ to $+\infty$ by a new matrix $T(u)$ corresponding to the transition $(x_0, +\infty, -\infty, x_0)$ where x_0 is some fixed point. In contrast with $\mathbf{T}(u)$ the matrix $T(u)$ satisfies the same commutation relations as the monodromy matrix for a finite interval [13]

$$R(u_1-u_2)T^{(1)}(u_1)T^{(2)}(u_2) = T^{(2)}(u_2)T^{(1)}(u_1)R(u_1-u_2) \tag{3}$$

where $T^{(1)} \equiv T \otimes \mathrm{id}$, $T^{(2)} = \mathrm{id} \otimes T$. It is remarkable that the algebra (3) is determined only by the properties of the local L-operator and does not depend on those of the physical vacuum. The choice of a definite physical vacuum corresponds to the choice of some representation of the algebra (3). The representations of (3) connected with the infinite-volume systems have a number of specific properties. These are, first, the existence of cuts in the plane of the complex argument u of the function $T(u)$ and, second, the vanishing of the quantum determinant of the matrix $T(u)$.

The problem of finding the excitation spectrum of the system reduces to that of determining the common spectrum of a commutative subalgebra $t(u)$ of the algebra (3). To solve the latter problem one cannot, generally speaking, use the so-called algebraic Bethe ansatz method [1, 2] since it works only over the ferromagnetic vacuum. That is why we use an alternative method, the functional Bethe ansatz (FBA) proposed in [5]. The above-mentioned peculiarities of the representations of (3) for the infinite-volume systems lead to noticeable modifications of the FBA scheme in comparison with [5].

In the present paper we realise our program for the simplest integrable model, the non-linear Schrödinger equation (NLS), repulsive case, zero density. The details of the calculations are, most often, omitted. A more extensive text will be published separately.

2. New definition of $\mathbf{T}(u)$

The model in question is described in terms of the pair of canonical fields $[\Psi(x), \Psi^*(y)] = \delta(x-y)$. The quantum state space is the Fock space with the vacuum defined by $\Psi(x)\,|0\rangle = 0$. Consider the L-operator $\partial/\partial x - \mathscr{L}(u,x)$

$$\mathscr{L}(u,x) = \begin{pmatrix} -\mathrm{i}u/2 & \sqrt{c}\,\Psi^*(x) \\ \sqrt{c}\,\Psi(x) & \mathrm{i}u/2 \end{pmatrix}$$

where $c > 0$ is the coupling constant. The monodromy matrix $\mathscr{T}_{x_-}^{x_+}(u)$ for the interval $[x_-, x_+]$ is defined as

$$\mathscr{T}_{x_-}^{x_+}(u) =: \overrightarrow{\exp} \int_{x_-}^{x_+} \mathscr{L}(u,x)\,\mathrm{d}x :$$

where $\overrightarrow{\exp}$ is the ordered exponential [2, 3]. Let us fix a point x_0 and define the Jost matrices by

$$\mathscr{T}_-(u) = \lim_{x_-\to-\infty} \mathscr{T}_{x_-}^{x_0}(u)\exp(-\tfrac{1}{2}\mathrm{i}u\sigma_3 x_-) \tag{4a}$$

$$\mathscr{T}_+(u) = \lim_{x_+\to+\infty} \exp(\tfrac{1}{2}\mathrm{i}u\sigma_3 x_+)\mathscr{T}_{x_0}^{x_+}(u). \tag{4b}$$

The matrix $\mathbf{T}(u)$ used in [2,3] is defined as

$$\mathbf{T}(u) = \mathscr{T}_+(u)\mathscr{T}_-(u).$$

The commutative family $\mathbf{A}(u)$, see (2), is known to serve as a generating function of the quantum integrals of motion and contains, in particular, the Hamiltonian

$$H = \int_{-\infty}^{+\infty} (\Psi_x^*\Psi_x + c\Psi^*\Psi^*\Psi\Psi)\, \mathrm{d}x.$$

Let $\boldsymbol{C}_+$, respectively $\boldsymbol{C}_-$, denote the upper (lower) complex half-plane $\mathrm{Im}\, u > 0$, respectively < 0. Let us introduce now the matrix $\mathscr{T}(u)$ by the formula

$$\mathscr{T}(u) = \mathscr{T}_-(u)P_\pm\mathscr{T}_+(u) = \begin{pmatrix} \mathscr{A}(u) & \mathscr{B}(u) \\ \mathscr{C}(u) & \mathscr{D}(u) \end{pmatrix} \qquad u \in \boldsymbol{C}_\pm \tag{5}$$

where the projection operators $P_\pm$ are defined as

$$P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

To understand better the origin of the formula (5) consider the finite volume case. Let $x_+ > x_0 > x_-$. The matrices $\mathscr{T}_{x_-}^{x_+}(u) = \mathscr{T}_{x_0}^{x_+}(u)\mathscr{T}_{x_-}^{x_0}(u)$ and $\widetilde{\mathscr{T}}(u) = \mathscr{T}_{x_-}^{x_0}(u)\mathscr{T}_{x_0}^{x_+}(u)$ have obviously the same trace $t(u)$ and both satisfy the commutation relation (3). After going to the limit $x_\pm \to \pm\infty$ and removing the exponential factors (4) the matrix $\mathscr{T}_{x_-}^{x_+}(u)$ goes over into $\mathbf{T}(u)$. Let us see what happens with $\widetilde{\mathscr{T}}(u)$ as $x_\pm \to \pm\infty$:

$$\widetilde{\mathscr{T}}(u) \simeq \mathscr{T}_-(u)\exp\{-\mathrm{i}u(x_+ - x_-)\sigma_3/2\}\mathscr{T}_+(u).$$

The factor $\exp\{-\mathrm{i}u(x_+ - x_-)\sigma_3/2\}$ behaves like $\exp\{\mp\mathrm{i}u(x_+ - x_-)/2\}P_\pm$ for $u \in \boldsymbol{C}_\pm$. Cancelling out the scalar factors $\exp\{\mp\mathrm{i}u(x_+ - x_-)/2\}$, which does not affect the relation (3), results in the formula (5) for $T(u)$.

The matrices $\mathscr{T}_{x_-}^{x_+}(u)$ and $\widetilde{\mathscr{T}}(u)$ contain the same information about the finite-volume system. This should also be true for the matrices $\mathbf{T}(u)$ and $\mathscr{T}(u)$ in the infinite-volume case. The matrix $\mathscr{T}(u)$ has, however, some advantages discussed already in the introduction.

Using the known properties of $\mathscr{T}_\pm(u)$ [3] one can establish the following properties of $\mathscr{T}(u)$.

(i) $\mathscr{T}(u)$ is an analytic function of u in the complex plane $\boldsymbol{C}$ except for the real axis $\boldsymbol{R}$ or, equivalently, is analytic for $u \in \boldsymbol{C}_\pm$.

(ii) $\mathscr{T}(u)$ satisfies the same relation (3) as the finite-volume monodromy matrix $\mathscr{T}_{x_-}^{x_+}(u)$. The R matrix in (3) is given by

$$R(u) = u - \mathrm{i}c\mathscr{P} \tag{6}$$

where $\mathscr{P}$ is the permutation operator in $\boldsymbol{C}^2 \otimes \boldsymbol{C}^2$. In other words, $\mathscr{T}(u)$ is a representation of the algebra defined by the relations (3).

(iii) The quantum determinant of $\mathscr{T}(u)$ vanishes

$$\mathscr{D}(u + \mathrm{i}c)\mathscr{A}(u) - \mathscr{B}(u + \mathrm{i}c)\mathscr{C}(u) = 0.$$

(iv) The trace $t(u) = \mathscr{A}(u) + \mathscr{D}(u)$ of $\mathscr{T}(u)$ equals $\mathbf{A}(u)$ for $u \in \boldsymbol{C}_+$ and $\mathbf{D}(u)$ for $u \in \boldsymbol{C}_-$ thus coinciding with the conventional generating function of integrals of motion [2, 3]. Note that $t(u)$ commute

$$[t(u_1), t(u_2)] = 0.$$

(v) Let * stand for the Hermitian conjugate with respect to the quantum space. Then

$$\mathscr{T}^*(u) = \sigma_1 \mathscr{T}(\bar{u}) \sigma_1.$$

(vi) The following equalities are valid:

$$\begin{array}{lll} \mathscr{C}(u)\,|0\rangle = 0 & & \\ \mathscr{A}(u)\,|0\rangle = |0\rangle & \mathscr{D}(u)\,|0\rangle = 0 & \text{for } u \in \boldsymbol{C}_+; \\ \mathscr{A}(u)\,|0\rangle = 0 & \mathscr{D}(u)\,|0\rangle = |0\rangle & \text{for } u \in \boldsymbol{C}_-. \end{array}$$

The eigenvectors of the operators $t(u)$ can be constructed in the spirit of the algebraic Bethe ansatz [1, 2] using $\mathscr{B}(u)$ as the creation operators. In the finite-volume case $\mathscr{B}(u)$ is a holomorphic function of u and the eigenvectors are constructed in the form $\mathscr{B}(\lambda_1)\ldots\mathscr{B}(\lambda_N)\,|0\rangle$, λ_n being real for $c > 0$. In the infinite-volume case the situation is more complicated. Since the function $\mathscr{B}(u)$ has the cut along the real axis, one must distinguish the values $\lambda_n \pm \mathrm{i}0$ for real λ_n. An eigenstate is then constructed as a sum over all possible choices of signs $\lambda_n \pm \mathrm{i}0$ in the product $\mathscr{B}(\lambda_1)\ldots\mathscr{B}(\lambda_N)$.

More precisely, let Λ be a finite set of real numbers and (Λ_+, Λ_-) be its arbitrary partition into two disjoint subsets. The multiparticle eigenstate $|\Lambda\rangle$ parametrised by Λ is

$$|\Lambda\rangle = \sum_{(\Lambda_+,\Lambda_-)} \prod_{\lambda_\pm \in \Lambda_\pm} \frac{\lambda_+ - \lambda_- - \mathrm{i}c}{\lambda_+ - \lambda_- + \mathrm{i}c}\,|\Lambda_+, \Lambda_-\rangle \tag{7}$$

$$|\Lambda_+, \Lambda_-\rangle = \left(\prod_{\lambda_+ \in \Lambda_-} \mathscr{B}(\lambda_+ + \mathrm{i}0)\right)\left(\prod_{\lambda_- \in \Lambda_-} \mathscr{B}(\lambda_- - \mathrm{i}0)\right)|0\rangle .$$

The corresponding eigenvalue $\tau(u)$ of $t(u)$ is

$$\tau(u) = \prod_{\lambda \in \Lambda} \frac{u - \lambda \pm \mathrm{i}c}{u - \lambda} \qquad u \in \boldsymbol{C}_\pm \tag{8}$$

and coincides with the well known result [3] for the eigenvalues of $\mathbf{A}(u)$.

One can easily verify the statements made using the commutation relations (3) and the property (vi) of $\mathscr{T}(u)$.

The drawback of this method of constructing the eigenvectors of $t(u)$ is that it uses essentially the property (vi) of $\mathscr{T}(u)$ which is satisfied for a limited number of models. Bearing in the mind possible applications to such models as sinh-Gordon and NLS of finite density, we shall present an alternative and more general method to diagonalise $t(u)$ which is free of the above restrictions.

3. Classical case

First of all, let us replace the matrix $\mathscr{T}(u)$ by the matrix $T(u)$ by means of the similarity transformation

$$T(u) = U^{-1}\mathscr{T}(u)U = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \qquad U = \begin{pmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix}. \tag{9}$$

The matrix U performs the isomorphism of Lie algebras $U^{-1}(.)U : \mathrm{su}(1,1) \to \mathrm{sl}(2,\boldsymbol{R})$. The matrix $T(u)$ has the same properties (i)–(iv) as $\mathscr{T}(u)$. The property (v) is replaced by the sl(2, $\boldsymbol{R}$)-type relation

$$T^*(u) = T(\bar{u}). \tag{10}$$

The property (vi) does not hold for $T(u)$, not that we shall use it.

Now we are going to investigate the representation $T(u)$ of the algebra (3) using the method proposed in [5] which we shall call the functional Bethe ansatz (FBA). The main idea of FBA is to realise the space of quantum states as some space of functionals on the joint spectrum of the commuting operators $B(u)$. One calculates then the explicit form of the operators $t(u)$ in such a B-representation which allows us usually to perform separation of variables for the eigenfunctions of $t(u)$.

To get a hint of the possible structure of the representation $T(u)$ let us consider first the classical case. The classical matrix $T(u)$ is constructed in the same way as the quantum one and has the analogous properties (i)–(v), the property (ii) being replaced by the Poisson bracket relation,

$$\{T^{(1)}(u_1), T^{(2)}(u_2)\} = [r(u_1 - u_2), T^{(1)}(u_1)T^{(2)}(u_2)] \tag{11}$$

where $r(u) = -c\mathscr{P}/u$, and the quantum determinant in (iii) being replaced by the ordinary determinant of $T(u)$. In addition, the asymptotic relation is valid for $T(u)$

$$T(u) \to U^{-1}P_{\pm}U = \frac{1}{2}\begin{pmatrix} 1 & \mp\mathrm{i} \\ \pm\mathrm{i} & 1 \end{pmatrix} \qquad u \to \pm\mathrm{i}\infty. \tag{12}$$

Let us define now the variables $p(\lambda)$, $q(\lambda)$ for real λ by the following equations:

$$\mathrm{e}^{-\mathrm{i}\pi q(\lambda)}B(\lambda + \mathrm{i}0) + \mathrm{e}^{\mathrm{i}\pi q(\lambda)}B(\lambda - \mathrm{i}0) = 0 \tag{13a}$$

$$\mathrm{e}^{-\mathrm{i}\pi q(\lambda)}A(\lambda + \mathrm{i}0) + \mathrm{e}^{\mathrm{i}\pi q(\lambda)}A(\lambda - \mathrm{i}0) = \mathrm{e}^{-p(\lambda)} \tag{13b}$$

$$\mathrm{e}^{-\mathrm{i}\pi q(\lambda)}D(\lambda + \mathrm{i}0) + \mathrm{e}^{\mathrm{i}\pi q(\lambda)}D(\lambda - \mathrm{i}0) = \mathrm{e}^{p(\lambda)}. \tag{13c}$$

To determine $p(\lambda)$ and $q(\lambda)$ uniquely we shall select the real solutions of (13). The existence of such solutions follows from the conjugation condition (10) and the absence of complex and real zeros of the function $B(u)$ due to the self-adjointness of the corresponding boundary problem for the Dirac operator $\partial/\partial x - U^{-1}\mathscr{L}(u,x)U$. The equations (13$b$) and (13$c$) for $p(\lambda)$ are equivalent by virtue of the identity $\det \mathbf{T}(\lambda) = 1$.

The quantities $p(\lambda)$ and $q(\lambda)$ are real, vanish as $|x| \to \infty$ and have canonical Poisson brackets

$$\{p(\lambda), p(\mu)\} = \{q(\lambda), q(\mu)\} = 0$$
$$\{p(\lambda), q(\mu)\} = c\delta(\lambda - \mu)$$

which are easily calculated from (13) and (11). The variables $p(\lambda)$ and $q(\lambda)$ are nothing else than the continuous analogue of the variables p_n, q_n introduced by Flaschka and McLaughlin [6] in the finite-volume case. Considering (13) as boundary value problems for the analytic functions $B(u)$, $A(u)$, $D(u)$ one can express them in terms of $p(\lambda)$, $q(\lambda)$:

$$B(u) = \mp\frac{\mathrm{i}}{2}\exp\left(-\int_{-\infty}^{+\infty}\frac{\mathrm{d}\lambda}{u-\lambda}q(\lambda)\right) \tag{14a}$$

$$A(u) = B(u)\left(\pm\mathrm{i} - \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\lambda}{u-\lambda}(G_+(\lambda) - 1)\right) \tag{14b}$$

$$D(u) = B(u)\left(\pm\mathrm{i} - \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\lambda}{u-\lambda}(G_-(\lambda) - 1)\right) \tag{14c}$$

where $u \in C_\pm$ and $G_\pm(\lambda)$ is

$$G_\pm(\lambda) = \exp\left(\mp p(\lambda) + \mathrm{P}\int_{-\infty}^{+\infty} \frac{\mathrm{d}\mu}{\lambda-\mu} q(\mu)\right)$$

with P standing for the principal value regularisation of $1/(\lambda-\mu)$. One can also express $G_\pm(\lambda)$ using the variable $\partial(\lambda) = \mathrm{i}p(\lambda) + \pi q(\lambda)$ which is more convenient for quantisation

$$G_\pm(\lambda) = \exp\left(\pm \mathrm{i}\partial(\lambda) + \int_{-\infty}^{+\infty} \frac{\mathrm{d}\mu}{\lambda-\mu\pm\mathrm{i}0} q(\mu)\right). \tag{15}$$

The functions $G_+(\lambda)$ and $G_-(\lambda)$ deserve a closer look. One can show that they coincide with the spectral densities [7] for the Dirac operator $\partial/\partial x - U^{-1}\mathscr{L}(u,x)U$ on the semi-axis $[x_0, +\infty)$, respectively $(-\infty, x_0]$, with appropriate boundary conditions. The functions $G_\pm(\lambda)$ are real and tend to 1 as $|\lambda| \to \infty$. Using (14) one can define $G_\pm(\lambda)$ also by

$$G_+ = \left[B^{-1}(\lambda+\mathrm{i}0)A(\lambda+\mathrm{i}0) - B^{-1}(\lambda-\mathrm{i}0)A(\lambda-\mathrm{i}0)\right]/2\mathrm{i} \tag{16a}$$

$$G_- = \left[B^{-1}(\lambda+\mathrm{i}0)D(\lambda+\mathrm{i}0) - B^{-1}(\lambda-\mathrm{i}0)D(\lambda-\mathrm{i}0)\right]/2\mathrm{i}. \tag{16b}$$

The Poisson brackets

$$\{B(u), B(v)\} = 0 \tag{17a}$$

$$\{G_\pm(\lambda), B(u)\} = \mp\frac{c}{\lambda-u} G_\pm(\lambda)B(u) \tag{17b}$$

$$\{G_\pm(\lambda), G_\pm(\mu)\} = \pm\mathrm{P}\frac{2c}{\lambda-\mu} G_\pm(\lambda)G_\pm(\mu) \tag{17c}$$

$$\{G_\pm(\lambda), G_\mp(\mu)\} = 0 \tag{17d}$$

then follow immediately from (11).

4. Quantisation

Let us return now to the quantum case. According to the FBA scenario described in the previous section the first problem is to find the joint spectrum of the commutative operator family $B(u)$. Since the direct spectral analysis of $B(u)$ seems, however, to be a rather hard task, we shall try instead to guess the answer. Let us conjecture that in the quantum case the representation (14*a*) for $B(u)$ is still valid, $q(\lambda)$ being now a commuting family of self-adjoint operators. Let us conjecture also that the joint spectrum of $q(\lambda)$ is simple, that is, $q(\lambda)$ can be realised as multiplication operators in the Hilbert space of functionals $V[q(\lambda)]$ of real-valued functions $q(\lambda)$. The functionals $V[q(\lambda)]$ are supposed to be square integrable with respect to some measure dm which is to be calculated later

$$\langle V, W\rangle = \int \overline{V}[q(\lambda)]\,W[q(\lambda)]\,\mathrm{d}m. \tag{18}$$

The next problem is to find how the operators $A(u)$ and $D(u)$ act on the functionals $V[q(\lambda)]$. Let us introduce first the quantum operators $G_\pm(\lambda)$ by the formulae (16) preserving the factor ordering shown there.

Applying to the first term in (16*a*) the identity

$$B^{-1}(\lambda)A(\lambda) = A(\lambda+\mathrm{i}c)B^{-1}(\lambda+\mathrm{i}c)$$

which follows from (3) and using the analyticity properties of $T(u)$ one concludes that $G_+(\lambda)$ is analytic in the strip $\mathrm{Im}\,\lambda \in (-c, 0)$. Quite analogously, $G_-(\lambda)$ is analytic for $\mathrm{Im}\,\lambda \in (0, c)$. Combining (10), (3) and (16) one deduces also the conjugation relations

$$B^*(\lambda) = B(\bar{\lambda}) \tag{19a}$$

$$G^*_\pm(\lambda) = G_\pm(\bar{\lambda} \mp \mathrm{i}c). \tag{19b}$$

The operators $B(u)$ and $G_\pm(\lambda)$ commute as follows:

$$B(u)B(v) = B(v)B(u) \tag{20a}$$

$$G_\pm(\lambda)B(u) = \frac{\lambda - u \pm \mathrm{i}c}{\lambda - u} B(u)G_\pm(\lambda) \tag{20b}$$

$$G_\pm(\lambda)G_\pm(\mu) = \frac{\lambda - \mu \mp \mathrm{i}c}{\lambda - \mu \pm \mathrm{i}c} G_\pm(\mu)G_\pm(\lambda) \tag{20c}$$

$$G_\pm(\lambda)G_\mp(\mu) = G_\mp(\mu)G_\pm(\lambda). \tag{20d}$$

The calculation leading to (20) is quite straightforward though rather long. It uses (3) and, in case of (20*d*), the property (iii) of $T(u)$.

Slightly modifying the classical formula (15) for $G_\pm(\lambda)$ one obtains the quantum operator $G_\pm(\lambda)$ acting on the functionals $V[q(\lambda)]$

$$G_\pm(\lambda) = \exp\left(\int_{-\infty}^{+\infty} \frac{\mathrm{d}\mu}{\lambda - \mu \pm \mathrm{i}0} q(\mu)\right) \exp\left(\int_{\lambda}^{\lambda \pm \mathrm{i}c} \frac{\delta}{\delta q(u)}\,\mathrm{d}u\right) \tag{21}$$

which can be shown to satisfy the commutation relations (20) with the operator $B(u)$ given by (14*a*). For the integral from λ to $\lambda \pm \mathrm{i}c$ in (21) to be defined correctly, it is sufficient to suppose that the functions $q(\lambda)$ are analytic in the strip $|\mathrm{Im}\,\lambda| < c + \varepsilon$ for some $\varepsilon > 0$.

Now we are able to determine the measure dm (18) from the conjugation condition (19*b*). It is most natural to look for the measure dm in the class of Gaussian measures. So, let dm be given by the continual integral

$$\int (\ldots)\,\mathrm{d}m = \frac{1}{\mathcal{N}} \int \prod_\lambda \mathrm{d}q(\lambda)\,(\ldots) \exp\left(-\frac{1}{2} \int \int \mathrm{d}\mu\,\mathrm{d}\nu\, q(\mu)M(\mu,\nu)q(\nu)\right) \tag{22}$$

the kernel $M(\mu, \nu)$ being real, symmetric and positive definite. The normalisation constant $\mathcal{N}$ in (22) is defined by the condition $\int 1\,\mathrm{d}m = 1$.

Calculating the conjugate of $G_\pm(\lambda)$ given by (21) with respect to the Hermitian form (18) with dm given by (22) and using the condition (19*b*), one obtains the following equations for the kernel $M(\mu, \nu)$:

$$\int_\lambda^{\lambda \pm \mathrm{i}c} M(\mu,\nu)\,\mathrm{d}\mu = \varepsilon_\mp(\lambda - \nu \pm \mathrm{i}c) - \varepsilon_\pm(\lambda - \nu) \tag{23}$$

where $\varepsilon_\pm(\lambda) \equiv (\lambda \pm \mathrm{i}0)^{-1}$. The shift by i$c$ in $\varepsilon_\mp$ should be understood as

$$\varepsilon_\mp(\lambda \pm \mathrm{i}c) = (\lambda \pm \mathrm{i}c)^{-1} \pm \pi \mathrm{i}\delta(\lambda \pm \mathrm{i}c).$$

The distribution $\delta(\lambda \pm \mathrm{i}c)$ is well defined as a linear functional on the functions of the real variable λ analytic in the strip $|\mathrm{Im}\,\lambda| < c + \varepsilon$.

Both equations (23) are equivalent since M is real. Being convolution equations, they are easily solved by the Fourier transform. The result is

$$M(\mu, \nu) \equiv M(\mu - \nu) \qquad M(\mu) = -\frac{1}{\mu^2} - \frac{\pi^2}{c^2}\frac{1}{\sinh^2(\pi/c)\mu}. \tag{24}$$

The singularity $-2/\mu^2$ in (24) is regularised in the standard manner

$$\int_{-\infty}^{+\infty} \frac{\varphi(\mu)}{\mu^2}\,\mathrm{d}\mu \equiv \int_0^{+\infty} \frac{\varphi(\mu) + \varphi(-\mu) - 2\varphi(0)}{\mu^2}\,\mathrm{d}\mu.$$

The kernel $M(\mu - \nu)$ is obviously real and symmetric. Its positivity follows from the positivity of the Fourier transform

$$\widetilde{M}(k) \equiv \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}k\mu} M(\mu)\,\mathrm{d}\mu = 2\pi\frac{|k|}{1 - \mathrm{e}^{-c|k|}} > 0.$$

For readers' convenience let us present also the covariance kernel $\mathcal{M}(\mu - \nu) \equiv \int q(\mu)q(\nu)\,\mathrm{d}m$ which is inverse to $M(\mu - \nu)$:

$$\widetilde{\mathcal{M}}(k) \equiv 1/M(k) = \frac{1}{2\pi}\frac{1 - \mathrm{e}^{-c|k|}}{|k|}$$

$$\mathcal{M}(k) \equiv \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}k\mu}\widetilde{\mathcal{M}}(k)\,\mathrm{d}k = \frac{1}{4\pi^2}\ln\left(1 + \frac{c^2}{\mu^2}\right).$$

As a result, we have constructed some representation of the algebra (20) with the involution (19). An equivalent description of the same representation can be given in terms of the creation–annihilation operators.

Let us introduce the fields $\Phi(\lambda)$, $\Phi^*(\lambda)$ by the formulae

$$\Phi(\lambda) = \int_{-\infty}^{+\infty} \mathrm{d}\mu\,\mathcal{M}(\lambda - \mu)\frac{\delta}{\delta q(\mu)} \tag{25a}$$

$$\Phi^*(\lambda) = q(\lambda) - \Phi(\lambda). \tag{25b}$$

It is easy to see that $\Phi(\lambda)$ and $\Phi^*(\lambda)$ are conjugate with respect to the scalar product (19), (22), (24) and belong to the Fock representation of CCR

$$[\Phi(\lambda), \Phi(\mu)] = [\Phi^*(\lambda), \Phi^*(\mu)] = 0$$

$$[\Phi(\lambda), \Phi^*(\mu)] = \mathcal{M}(\lambda - \mu)$$

having the vacuum $|0\rangle$ $[q(\lambda)] \equiv 1$, $\Phi(\lambda)\,|0\rangle = 0$.

Let us introduce also the operators

$$F_{\pm}(\lambda) = \exp\left(-\int_{-\infty}^{+\infty} \varepsilon_{\pm}(\lambda - \mu)\Phi(\mu)\,\mathrm{d}\mu\right) \tag{26a}$$

$$F_{\pm}^*(\lambda) = \exp\left(-\int_{-\infty}^{+\infty} \varepsilon_{\pm}(\lambda - \mu)\Phi^*(\mu)\,\mathrm{d}\mu\right) \tag{26b}$$

commuting as follows

$$[F_\pm(\lambda), F_\pm^*(\mu)] = 0 \tag{27a}$$

$$F_\pm(\lambda)F_\mp^*(\mu) = \frac{\lambda-\mu\pm \mathrm{i}c}{\lambda-\mu\pm \mathrm{i}0} F_\mp^*(\mu)F_\pm(\lambda). \tag{27b}$$

Using (14*a*), (21), (25), (26) one can express B and G in terms of F and F^*:

$$B(\lambda\pm \mathrm{i}0) = \mp\tfrac{1}{2}\mathrm{i}F_\pm^*(\lambda)F_\pm(\lambda) \tag{28a}$$

$$G\pm(\lambda) = [F_\pm^*(\lambda)]^{-1}F_\mp^{-1}(\lambda\pm \mathrm{i}c). \tag{28b}$$

Let us define now the operators $A(u)$ and $D(u)$ by the formulae (14*b*) and (14*c*) preserving the factor ordering shown there and using for $B(u)$ and $G_\pm(\lambda)$ the representations constructed above. The operator $C(u)$ is then defined as $C(u) = B(u+\mathrm{i}c)^{-1}D(u+\mathrm{i}c)A(u)$.

A straightforward, though rather long, calculation shows that the matrix $T(u)$ thus constructed has all the necessary properties (i)–(v) . Unfortunately, the conditions (i)–(v) do not define $T(u)$ uniquely. The situation is the same as in case of representations of CCR with infinitely many degrees of freedom.

To sum up, we have constructed two representations of the algebra (3) with the involution (10). The first one is the original representation (9) in terms of the fields Ψ, Ψ^*. The second one is that built in terms of Φ, Φ^* using (26), (28). There are good reasons to suppose these two representations to be isomorphic.

First, this is true in the classical limit $c\to 0$ as we have seen in the previous section.

Second, the vacuum expectations $\langle 0|\, B(u_1)\ldots B(u_N)\,|0\rangle$ can be calculated for both representations and have the same value. For the *a priori* built representation of $T(u)$ the calculation is made either with the use of (14*a*) and taking the Gaussian integral (22) or with the use of (28*a*) and of the commutation relations (27). In the case of the original representation $T(u)$ in terms of Ψ, Ψ^* the calculation is rather cumbersome. It is based on the technique due to Korepin [8] and uses the relation $2\mathrm{i}B(u) = (\mathscr{A}-\mathscr{B}+\mathscr{C}-\mathscr{D})(u)$ and the properties (ii) and (vi) of $\mathscr{T}(u)$.

The result is the same for both representations and can be formulated as follows. Let $X_\pm$ be finite subsets of $\mathbf{C}_\pm$, respectively, and $|X|$ denote the cardinality of X. Then

$$\langle X\rangle \equiv \langle 0|\prod_{\xi\in X} B(\xi)\,|0\rangle = (\tfrac{1}{2}\mathrm{i})^{|X|}(-1)^{|X_+|}\prod_{\xi_\pm\in X_\pm}\frac{\xi_+-\xi_-+\mathrm{i}c}{\xi_+-\xi_-}.$$

The third test of our hypothesis consists in calculating the eigenstates and eigenvalues of $t(u)$ using (14*b,c*) and (28). To describe the result we shall use the same notation as in (7). The eigenstates corresponding to the eigenvalues (8) of $t(u)$ are

$$|\Lambda\rangle = \sum_{(\Lambda_+,\Lambda_-)}(-1)^{|\Lambda_+|}\prod_{\lambda_\pm\in\Lambda_\pm}\frac{\lambda_+-\lambda_--\mathrm{i}c}{\lambda_+-\lambda_-+\mathrm{i}0}\,|\Lambda_+,\Lambda_-\rangle \tag{29}$$

where

$$|\Lambda_+,\Lambda_-\rangle = \left(\prod_{\lambda_+\in\Lambda_+}F_+^*(\lambda_+)\right)\left(\prod_{\lambda_-\in\Lambda_-}F_-^*(\lambda_-)\right)|0\rangle\,.$$

Note that the states $|\Lambda\rangle$ in (29) differ from those in (7) by some scalar factors.

5. Discussion

Using the new definition of the monodromy matrix for the infinite interval we have succeeded in reproducing the known results concerning the spectrum of the NLS model. An advantage of our method as compared to the traditional ones is that it does not use the existence of ferromagnetic vacuum,or the highest vector ω of $T(u)$, satisfying $C(u)\omega = 0$. This circumstance allows us to easily apply our method to the models with non-ferromagnetic vacuum. In particular, replacing the R matrix (6) with the XXZ model R matrix one can investigate the sinh-Gordon model [9]. It is challenging problem to construct the matrix $T(u)$ corresponding to the NLS model of finite density and temperature or to the Toda chain. It would be interesting also to generalise the method to the real forms of sl(2, $\boldsymbol{C}$) other then sl(2, $\boldsymbol{R}$).

The crucial problem is the identification of the *a priori* constructed representations $T(u)$ of the algebra (3) with concrete integrable models. In this connection, a kind of the quantum Gelfand–Levitan equation would be useful which could help to express the local fields like $\Psi(x)$, $\Psi^*(x)$ in terms of $G_\pm(\lambda)$.

There are also a lot of purely mathematical problems concerning the rigorous justification of all the conjectures made. In particular, it would be interesting to prove directly the completeness of the eigenstates (29).

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